A correction to: Stochastic and nonlinear forcing of wavepackets in a Mach 0.9 jet

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This note contains a correction to equation (16) in AIAA Paper 2015-2217: "Stochastic and nonlinear forcing of wavepackets in a Mach 0.9 jet," as well as some additional clarification of the empirical resolvent mode (ERM) method described in the same paper. The notation and terminaology used in this note follows that of the paper. Before turning attention to ERM, it's useful to write proper orthogonal decomposition (POD) as an optimization problem in a form similar to that used later for ERM.

I. Proper orthogonal decomposition

The POD modes of the output data Y are the columns of $\tilde{Y}_{POD} = Y\Psi_{POD}$. If we write the expansion coefficient matrix as

$$\Psi_{POD} = \begin{bmatrix} \psi_{POD}^{(1)} & \psi_{POD}^{(2)} & \cdots & \psi_{POD}^{(N_s)} \end{bmatrix}, \tag{1}$$

then the vector of coefficients that define the first POD mode is the solution of the optimization problem

$$\psi_{POD}^{(1)} = \underset{\|\psi\|=1}{\arg\max} \|Y\psi\|^2 = \underset{\psi}{\arg\max} \frac{\psi^* Y^* Y \psi}{\psi^* \psi}. \tag{2}$$

The additional expansion coefficients are then defined as

$$\psi_{POD}^{(k)} = \underset{\|\psi\|=1}{\arg\max} \|Y_k\psi\|^2 = \underset{\psi}{\arg\max} \frac{\psi^* Y_k^* Y_k \psi}{\psi^* \psi}, \tag{3}$$

where

$$Y_k = Y - \sum_{s=1}^{k-1} Y\psi_{POD}^{(s)}\kappa_{POD}^{(s)} \tag{4}$$

and

$$\kappa_{POD}^{(s)} = \left(\psi_{POD}^{(s)}\right)^*. \tag{5}$$

The $\kappa_{POD}^{(s)}$ vectors are coefficnets for reconstructing the data from the modes. Conceptually, Y_k is a modified data matrix in which all content within of range of the first $k-1$ modes has been removed such that it is orthogonal to these modes. Therefore, the optimization in equation (3) finds the most energetic mode that is orthogonal to all prevuios modes.

To solve the maximization problem, we seek the extremum of the function

$$J = \psi^* Y_k^* Y_k \psi - \lambda\left(\psi^*\psi - 1\right), \tag{6}$$

which are obtained when

$$\frac{\partial J}{\partial \psi} = 2\left(Y_k^* Y_k \psi - \lambda\psi\right) = 0. \tag{7}$$

We therefore have the eigenvalue problem

$$Y_k^* Y_k \psi_{POD}^{(k)} = \lambda_{POD}^{(k)} \psi_{POD}^{(k)}. \tag{8}$$

The global extremum is given by the eigenvector corresponding to the largest eigenvalue. Since the matrix $Y_k^* Y_k$ is Hermitian, it's eigenvectors are orthogonal and we can combine the eigenvalue problem for every k into the single eigenvalue problem

$$Y^* Y \Psi_{POD} = \Psi_{POD} \Lambda_{POD}, \tag{9}$$

which is equivalent to equation (14) in the paper.

II. Empirical resolvent modes

We now describe ERM using the approach outlined above. To begin, write the ERM expansion coefficnet matrix as

$$\Psi_{ERM} = \left[\psi_{ERM}^{(1)} \quad \psi_{ERM}^{(2)} \quad \cdots \quad \psi_{ERM}^{(N_s)} \right]. \tag{10}$$

The coefficient vectors are the solutions of

$$\psi_{ERM}^{(1)} = \arg\max_{\psi} \frac{\|Y\psi\|^2}{\|F\psi\|^2} = \arg\max_{\psi} \frac{\psi^* Y^* Y \psi}{\psi^* F^* F \psi}. \tag{11}$$

and

$$\psi_{ERM}^{(k)} = \arg\max_{\psi} \frac{\|Y_k\psi\|^2}{\|F_k\psi\|^2} = \arg\max_{\psi} \frac{\psi^* Y_k^* Y_k \psi}{\psi^* F^* F \psi}, \tag{12}$$

where

$$Y_k = Y - \sum_{s=1}^{k-1} Y \psi_{ERM}^{(s)} \kappa_{ERM}^{(s)}, \tag{13}$$

$$F_k = F - \sum_{s=1}^{k-1} F \psi_{ERM}^{(s)} \kappa_{ERM}^{(s)}, \tag{14}$$

and

$$\kappa_{ERM}^{(s)} = \left(\psi_{ERM}^{(s)} \right)^* F^* F. \tag{15}$$

The modified data and force matrices have the same meaning as before, and the $\kappa_{ERM}^{(s)}$ vectors are again coefficients for reconstructing the data from the modes. The resulting output and force modes are each orthogonal and maximize the gain between them since they maximize the arguments in equations (11) and (12).

Equations (10)-(15) together replace equation (16) in the paper. The problem with the original equation is that it incorrectly contains *matrix* norms of the modal matrices rather than vector norms on the individual modes. This error was made in an attempt to compactly represent the optimization in one expression.

To solve the maximization problem, we seek the extremum of the function

$$J = \psi^* Y_k^* Y_k \psi - \lambda \left(\psi^* F^* F \psi - 1 \right), \tag{16}$$

which is obtained when

$$\frac{\partial J}{\partial \psi} = 2 \left(Y_k^* Y_k \psi - \lambda \psi^* F^* F \psi \right) = 0. \tag{17}$$

We therefore have the generalized eigenvalue problem

$$Y_k^* Y_k \psi_{ERM}^{(k)} = \lambda_{ERM}^{(k)} F^* F \psi_{ERM}^{(k)}. \tag{18}$$

The global extremum is given by the eigenvector corresponding to the largest eigenvalue. Since the matrices $Y_k^* Y_k$ and $F_k^* F_k$ are both Hermitian, the eigenvectors are orthogonal and we can combine the eigenvalue problem for every k into the single generalized eigenvalue problem

$$Y^* Y \Psi_{ERM} = F^* F \Psi_{ERM} \Lambda_{ERM}. \tag{19}$$

Solving this eigenvalue problem is equivalent to performing the simultaneous diagonalization described in the paper. To see this, suppose that Λ_{ERM} simultaneously diagonalizes $Y^* Y$ and $F^* F$:

$$\Psi_{ERM}^* Y^* Y \Psi_{ERM} = \Lambda_{ERM}, \tag{20a}$$

$$\Psi_{ERM}^* F^* F \Psi_{ERM} = I. \tag{20b}$$

Then left-multiplying equation (19) by Ψ_{ERM}^* gives the identity $\Lambda_{ERM} = \Lambda_{ERM}$, so the generalized eigenvalue problem is satisfied.